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Coherent Orthogonal Polynomials

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Abstract

We discuss as a fundamental characteristic of orthogonal polynomials like the existence of a Lie algebra behind them, can be added to their other relevant aspects. At the basis of the complete framework for orthogonal polynomials we put thus –in addition to differential equations, recurrence relations, Hilbert spaces and square integrable functions– Lie algebra theory.

We start here from the square integrable functions on the open connected subset of the real line whose bases are related to orthogonal polynomials. All these one-dimensional continuous spaces allow, besides the standard uncountable basis $\{|x\rangle\}$, for an alternative countable basis $\{|n\rangle\}$. The matrix elements that relate these two bases are essentially the orthogonal polynomials: Hermite polynomials for the line and Laguerre and Legendre polynomials for the half-line and the line interval, respectively.

Differential recurrence relations of orthogonal polynomials allow us to realize that they determine a unitary representation of a non-compact Lie algebra, whose second order Casimir \mathcal{C} gives rise to the second order differential equation that defines the corresponding family of orthogonal polynomials. Thus, the Weyl-Heisenberg algebra $\mathfrak{h}(1)$ with $\mathcal{C} = 0$ for Hermite polynomials and $\mathfrak{su}(1,1)$ with $\mathcal{C} = -1/4$ for Laguerre and Legendre polynomials are obtained.

Starting from the orthogonal polynomials the Lie algebra is extended both to the whole space of the \mathcal{L}^2 functions and to the corresponding Universal Enveloping Algebra and transformation group. Generalized coherent states from each vector in the space \mathcal{L}^2 and, in particular, generalized coherent polynomials are thus obtained.

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1 Introduction

Orthogonal polynomials are relevant in many fields of mathematics: differential equations, algebras and Lie groups, Hilbert spaces and generalized Fourier series are, perhaps, the principal ones.

They are essentially orthogonal bases of square integrable functions $\mathcal{L}^2(E)$. We consider in this paper the cases discussed by Cambanis in [1], where E is a connected open subset of the real line, i.e. $E = (a, b) \subset \mathbb{R}$ with $-\infty \leq a < b \leq +\infty$. We will deal with the Hermite functions when $E = (-\infty, +\infty)$, the Laguerre functions for $E = (a, +\infty)$ with $-\infty < a$ or $E = (-\infty, b)$ with $b < +\infty$ (both with an appropriate change of variables reduce to $E = (0, +\infty)$) and the Legendre polynomials for $E = (a, b)$ with $-\infty < a < b < +\infty$ (that, with a suitable change of variables, reduces to $E = (-1, +1)$). We will see that the operators acting on each vector space have a Lie algebra structure.

The framework we are dealing with is related to the intertwining of the basis that defines the space and the alternative countable basis introduced by the orthogonal polynomials. The generators of the Lie algebra are shown to be related to first order differential recurrence relations while \mathcal{C} , the second order invariant of the algebra, originates the second order differential equation that defines the family of orthogonal polynomials. From the generators, the full set of operators on the space $\mathcal{L}^2(E)$ is built inside the Universal Enveloping Algebra (UEA), i.e. the closure of the underlying vector space of a Lie algebra that has the ordered monomials constructed on the generators as a basis. In particular, as the group elements belong to the UEA, the group transformations on $\mathcal{L}^2(E)$ are in deep relationship with the orthogonal polynomials. Specifically, each type of orthogonal polynomials is related to a unitary irreducible representation (UIR) of a non-compact Lie group: Hermite functions (as it is well-known from the quantum harmonic oscillator) support a UIR of the Weyl-Heisenberg group $H(1)$ and the Laguerre and Legendre functions belong to the fundamental representation of the discrete series of $SU(1, 1)$.

As algebra and group operators are defined on the whole $\mathcal{L}^2(E)$ space we can obtain in an operatorial way peculiar combinations of vectors of $\mathcal{L}^2(E)$ as, for instance, generalized coherent orthogonal polynomials.

Our starting point is the separable Hilbert space $\mathcal{L}^2(E)$ equipped with the basis $\{|x\rangle\}$ on the connected open subset $E = (a, b) \subset \mathbb{R}$. Orthonormality and completeness are

$$\langle x | y \rangle = \delta(x - y), \quad \int_a^b |x\rangle dx \langle x| = \mathbb{I}. \quad (1)$$

For each type of orthogonal polynomials we will write first the vector space –essentially to fix the notations– and then we will discuss the operators acting on it.

2 Hermite polynomials

We start our discussion from the Hilbert space constructed by means of the Hermite polynomials $H_n(x)$, defined on the full real line $(a, b) = (-\infty, \infty)$, because in this case both, vector space and Lie algebra structure, are well known and we shall limit ourselves to put together, in a compact way, known results.

To begin with, some factors are attached to the $H_n(x)$, writing the usually called Hermite functions $K_n(x)$:

$$K_n(x) := \frac{e^{-x^2/2}}{\sqrt{2^n n!} \sqrt{\pi}} H_n(x), \quad n \in \mathbb{N}. \quad (2)$$

In terms of them orthonormality and completeness are [2]

$$\begin{aligned} \int_{-\infty}^{\infty} K_n(x) K_m(x) dx &= \delta_{n,m}, \\ \sum_{n=0}^{\infty} K_n(x) K_n(y) &= \delta(x - y). \end{aligned} \quad (3)$$

Hence, the functions $\{K_n(x)\}$ determine an orthonormal basis for the real square integrable functions on the line, $\mathcal{L}^2((-\infty, \infty)) = \mathcal{L}^2(\mathbb{R})$ [3]. We can now define the set $\{|n\rangle\}_{n \in \mathbb{N}}$:

$$|n\rangle := \int_{-\infty}^{\infty} |x\rangle K_n(x) dx, \quad (4)$$

that is isomorphic to $\{K_n(x)\}$ and also an orthonormal basis since from (1), (2) and (3)

$$\langle n|m\rangle = \delta_{n,m}, \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{I}.$$

Note that, instead of the usual heavy notation $|K_n(x)\rangle$, we use the physical notation $|n\rangle$ to emphasize the two bases $\{|x\rangle\}_{x \in \mathbb{R}}$ and $\{|n\rangle\}_{n \in \mathbb{N}}$ of the Hilbert space $\mathcal{L}^2(\mathbb{R})$. Orthogonal functions play the role of transition matrices and, as the space is real, are written as

$$K_n(x) = \langle x|n\rangle = \langle n|x\rangle.$$

Now, in complete analogy with the Fourier analysis on the circle, an arbitrary vector $|f\rangle$ of $\mathcal{L}^2(\mathbb{R})$ can be written

$$|f\rangle = \int_{-\infty}^{+\infty} |x\rangle dx \langle x|f\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|f\rangle, \quad (5)$$

where the wave function $f(x)$ and the sequence $\{f_n\}_{n \in \mathbb{N}}$ describe the vector $|f\rangle$ in the two bases:

$$f(x) := \langle x|f\rangle = \sum_{n=0}^{\infty} K_n(x) f_n \quad f_n := \langle n|f\rangle = \int_{-\infty}^{+\infty} K_n(x) f(x) dx. \quad (6)$$

In particular, the completeness of the two bases determines the inner product as well as the Parseval identity:

$$\langle g|f\rangle = \sum_{n=0}^{\infty} g_n \cdot f_n = \int_{-\infty}^{+\infty} g(x) \cdot f(x) dx, \quad \sum_{n=0}^{\infty} f_n^2 = \int_{-\infty}^{+\infty} f(x)^2 dx. \quad (7)$$

Let us introduce now the operators defined on the line that, as it is well-known, are functions of creation, annihilation and number operators. Following the factorization method [4, 5] we consider, among the recurrence relations of Hermite polynomials, those that include first order derivatives [2]:

$$H'_n(x) = 2n H_{n-1}(x), \quad H'_n(x) - 2x H_n(x) = -H_{n+1}(x). \quad (8)$$

The second order differential equation, that defines the Hermite polynomials, is obtained by subsequent applications of recurrence relations. However, the fundamental limitation of such an approach is that in the factorization method the problem has been considered from the point of view of differential equations theory and, hence, the labels are assumed to be parameters.

We shall show here that a consistent Hilbert space framework, where the operators are well defined, allows us to reformulate special functions in the Lie algebra representation scheme, such that, in particular, Hermite functions support a well defined UIR of the Weyl-Heisenberg algebra $\mathfrak{h}(1)$. Effectively, we can rewrite eqs.(8) in terms of Hermite functions obtaining

$$a K_n(x) = \sqrt{n} K_{n-1}(x), \quad a^\dagger K_n(x) = \sqrt{n+1} K_{n+1}(x), \quad (9)$$

where a and a^\dagger are the annihilation and creation operators of the harmonic oscillator:

$$a := \frac{1}{\sqrt{2}} (X + D_x), \quad a^\dagger := \frac{1}{\sqrt{2}} (X - D_x). \quad (10)$$

In addition to the two previous operators X and D_x

$$X f(x) := x f(x), \quad D_x f(x) := f'(x), \quad (11)$$

we introduce two other operators N and I defined by

$$N K_n(x) := n K_n(x), \quad I K_n(x) := K_n(x), \quad (12)$$

in order to complete our set of operators. Note that the change introduced by substituting the label n of the Hermite polynomials for the operator N is far to be irrelevant since N does not commute with a and a^\dagger :

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger.$$

The whole Weyl-Heisenberg algebra follows

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [a, a^\dagger] = I, \quad [I, \bullet] = 0.$$

Expression (4) allows us to connect to the operators a , a^\dagger , N and I , defined on the set $\{K_n(x)\}$, four related operators defined on the line Hilbert space that we denote with the same symbols:

$$a |n\rangle = \sqrt{n} |n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad N |n\rangle = n |n\rangle, \quad I |n\rangle = |n\rangle.$$

The quadratic invariant operator \mathcal{C} can be written [6]

$$\mathcal{C} = \{a, a^\dagger\} - 2(N + 1/2) I.$$

As, by inspection, we are dealing with the representation $\mathcal{C} = 0$ the Schrödinger equation of the quantum harmonic oscillator is obtained, by means of eqs.(9–12), i.e.

$$\mathcal{C} K_n(x) \equiv (X^2 - D_x^2 - (2N + 1)) K_n(x) = 0,$$

that, rewritten in terms of the Hermite polynomials allows us to recover the equation defining the Hermite polynomials

$$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0.$$

Thus, we can conclude that the Hermite functions constitute a basis of the UIR $\mathcal{C} = 0$ of the Weyl-Heisenberg algebra $h(1)$ and the whole UEA of $h(1)$ is defined on the $\{K_n(x)\}$. In particular, as all elements of the Weyl-Heisenberg group $H(1)$ are contained inside the UEA of $h(1)$, all the transformations of $H(1)$ can be realized in the space $\{K_n(x)\}$. As an example, eqs.(9) allow to obtain the simplest normalized coherent Hermite function as [7]:

$$K_z(x) := \frac{1}{\sqrt{e^{|z|^2}}} e^{\alpha a^\dagger} K_0(x) = \frac{1}{\sqrt{e^{|z|^2}}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} K_n(x), \quad z \in \mathbb{C}.$$

3 Laguerre polynomials

The Laguerre functions differ from Hermite ones in the relevant fact that recurrence relations depend not only on the variable x (as it happens for a and a^\dagger in eqs.(10)) but also on the parameter n .

The discussion is similar to the previous one. We shall start renormalizing the Laguerre polynomials $L_n(x)$ by defining in this way the Laguerre functions

$$M_n(x) := e^{-x/2} L_n(x).$$

Orthonormality and completeness relations are [1]:

$$\begin{aligned} \int_0^\infty M_n(x) M_m(x) dx &= \delta_{n,m}, \\ \sum_{n=0}^\infty M_n(x) M_n(y) &= \delta(x-y). \end{aligned} \tag{13}$$

In analogy with expression (4), expressions (1) allows us to define

$$|n\rangle := \int_0^\infty |x\rangle M_n(x) dx,$$

and to obtain a countable basis $\{|n\rangle\}_{n \in \mathbb{N}}$ of the half-line, in addition to the standard basis $\{|x\rangle\}_{x \in (0, \infty)}$. The functions $M_n(x)$ are the transformation matrix elements that relate the two orthonormal bases

$$M_n(x) = \langle x | n \rangle = \langle n | x \rangle,$$

and for an arbitrary vector $|f\rangle$ of the Hilbert space $\mathcal{L}^2((0, \infty))$ all formulas (5-7) are valid after the changes $(-\infty, +\infty) \rightarrow (0, +\infty)$ and $\{K_n(x)\} \rightarrow \{M_n(x)\}$.

In order to obtain the operators acting on $\mathcal{L}^2((0, \infty))$, let us write the recurrence formulas involving derivatives for the Laguerre polynomials [8]

$$x L'_n(x) - n L_n(x) = -n L_{n-1}(x), \quad x L'_n(x) + (1 + n - x) L_n(x) = (n + 1) L_{n+1}(x),$$

that can be rewritten on terms of the functions $M_n(x)$ in a quite more symmetrical form

$$\begin{aligned} J_- M_n(x) &:= \left(-x \frac{d}{dx} + n - \frac{x}{2} \right) M_n(x) = n M_{n-1}(x), \\ J_+ M_n(x) &:= \left(x \frac{d}{dx} + n + 1 - \frac{x}{2} \right) M_n(x) = (n + 1) M_{n+1}(x). \end{aligned} \tag{14}$$

Because the operator XD_x is zero for $x = 0$, we can properly define J_\pm as true operators on the half-line Hilbert space

$$J_- := -XD_x + N - \frac{X}{2}, \quad J_+ := XD_x + N + 1 - \frac{X}{2}. \tag{15}$$

Like in the Hermite case, the introduction of these full operatorial structures is not trivial since the operators N and J_\pm do not commute

$$[N, J_\pm] = \pm J_\pm.$$

The operators acting on the basis $\{|n\rangle\}$ are equivalent to the ones acting on the basis $\{M_n(x)\}$

$$J_- |n\rangle = n |n-1\rangle, \quad J_+ |n\rangle = (n+1) |n+1\rangle, \quad N |n\rangle = n |n\rangle. \tag{16}$$

Defining $J_3 := N + 1/2$, commutators and anticommutators of J_{\pm} are

$$[J_+, J_-] = -2J_3, \quad \{J_+, J_-\} = 2J_3^2 + 1/2.$$

and, $\{M_n(x)\}$ and $\{|n\rangle\}$ are found to be representations of the $su(1, 1)$ algebra:

$$[J_3, J_{\pm}] = \pm J_{\pm} \quad [J_+, J_-] = -2J_3.$$

The differential equation defining the Laguerre polynomials can be obtained, following the factorization method, from

$$(J_+ J_- - N^2) M_n(x) = 0,$$

or

$$(J_- J_+ - (N + 1)^2) M_n(x) = 0.$$

Both expressions, using eqs. (15), give

$$-X (X D_x^2 + D_x + N + 1/2 - X/4) M_n(x) = 0,$$

which is equivalent, since X is never the null operator on $(0, +\infty)$, to

$$(X D_x^2 + D_x + N + 1/2 - X/4) M_n(x) = 0, \quad (17)$$

that, rewritten in terms of the $\{L_n(x)\}$, is the definition equation of the Laguerre polynomials, i.e.

$$x L_n''(x) + (1 - x) L_n'(x) + n L_n(x) = 0.$$

Alternatively, following the algebraic approach, the anticommutator $\{J_+, J_-\}$ is related to the Casimir operator \mathcal{C} . From eqs.(16)

$$\mathcal{C} = \left(J_3^2 - \frac{1}{2} \{J_+, J_-\} \right) = -\frac{1}{4},$$

that, introducing expressions (15), gives the operational form of eq.(17), i.e. the definition of Laguerre polynomials.

Since $\mathcal{C} = -1/4$ and $J_{\pm}^{\dagger} = J_{\mp}$ we are dealing with the fundamental representation of the discrete series of UIR of $su(1, 1)$. Because $\mathcal{C} = k(k - 1)$ with k the maximum weight of the representation, we find $k = 1/2$ in agreement with the minimum eigenvalue 0 of the operator N of the Laguerre polynomials. Hence, the spectrum of the operator J_3 is $1/2, 3/2, 5/2, \dots$

Again, from the algebra $su(1, 1)$ we can move to its UEA and to the group $SU(1, 1)$. In this way, a coherent Laguerre polynomial, $L_{\alpha}(x)$, could thus be defined à la Perelomov [7]. These coherent states are in correspondence with the points of the upper sheet of the two-sheet hyperboloid $\mathbb{H} = \{(y_0, y_1, y_2) \mid y_0^2 - y_1^2 - y_2^2 = 1, y_0 > 0\}$, and we can parametrize

the points of \mathbb{H} by the coordinates (ξ, θ) , with $\xi \in \mathbb{R}^+$ and $\theta \in [0, 2\pi)$, as follows: $y_0 = \cosh \xi$, $y_1 = \sinh \xi \cos \theta$, $y_2 = \sinh \xi \sin \theta$. Then

$$\begin{aligned} L_\alpha(x) &:= e^{x/2} e^{\hat{\xi}J_+ - \hat{\xi}^*J_-} M_0(x) = e^{x/2} e^{\alpha J_+} (1 - |\alpha|^2)^{J_3} e^{-\alpha^* J_-} M_0(x) \\ &= e^{x/2} (1 - |\alpha|^2)^{1/2} e^{\alpha J_+} M_0(x) \\ &= (1 - |\alpha|^2)^{1/2} \sum_{n=0}^{\infty} \alpha^n L_n(x), \end{aligned} \tag{18}$$

where $\hat{\xi} = \xi e^{i\theta}$ and $\alpha = e^{i\theta} \tanh \xi$.

4 Legendre polynomials

The approach is similar to that of Laguerre polynomials, with the difference that the interval $E = (a, b) \subset \mathbb{R}$ is now $(-1, 1)$. We have [2]:

$$\begin{aligned} \int_{-1}^1 P_n(x) (n + 1/2) P_m(x) dx &= \delta_{n,m}, \\ \sum_{n=0}^{\infty} P_n(x) (n + 1/2) P_n(y) &= \delta(x - y). \end{aligned} \tag{19}$$

As before expressions (19) allow us to define a new basis $\{|n\rangle\}_{n \in \mathbb{N}}$ in the interval $(-1, 1)$ in addition to the standard basis $\{|x\rangle\}_{x \in (-1, 1)}$:

$$|n\rangle := \int_{-1}^{+1} |x\rangle \sqrt{n + 1/2} P_n(x) dx.$$

The $P_n(x)$ polynomials are thus the transformation matrix elements that relate both two orthonormal bases:

$$\sqrt{n + 1/2} P_n(x) = \langle x | n \rangle = \langle n | x \rangle,$$

and, in complete analogy with the previous cases, for an arbitrary vector $|f\rangle$ of the Hilbert space $\mathcal{L}^2((-1, +1))$ all formulas (5-7) are valid for the Legendre case after the changes $(-\infty, +\infty) \rightarrow (-1, +1)$ and $\{K_n(x)\} \rightarrow \{\sqrt{n + 1/2} P_n(x)\}$.

The vector space structure described, let us now study to the operators. The recurrence formulae [8]:

$$\begin{aligned} (x^2 - 1) P'_n(x) &= (n + 1) (P_{n+1}(x) - x P_n(x)), \\ (x^2 - 1) P'_n(x) &= n (x P_n(x) - P_{n-1}(x)), \end{aligned} \tag{20}$$

can be rewritten in terms of differential operators as follows:

$$\begin{aligned} J_- P_n(x) &\equiv ((1 - X^2)D_x + X N) P_n(x) = n P_{n-1}(x), \\ J_+ P_n(x) &\equiv (-(1 - X^2)D_x + X (N + 1)) P_n(x) = (n + 1) P_{n+1}(x). \end{aligned} \tag{21}$$

Again, the change we introduced into eqs. (21), is not irrelevant since

$$[N, J_{\pm}] = \pm J_{\pm},$$

and, also in this case, the hermiticity relation $(J_{\pm})^{\dagger} = J_{\mp}$ (and, thus, the unitarity of the representation) is imposed by the recurrence formulas.

The action of the operators J_{\mp} on the vectors $|n\rangle$ is from (21)

$$J_- |n\rangle = n |n-1\rangle, \quad J_+ |n\rangle = (n+1) |n+1\rangle. \quad (22)$$

Finally from (22) we obtain:

$$\begin{aligned} J_+ J_- - N^2 &= -(1 - X^2) ((1 - X^2) D_x^2 - 2X D_x + N(N+1)) \equiv 0, \\ J_- J_+ - (N+1)^2 &= -(1 - X^2) ((1 - X^2) D_x^2 - 2X D_x + N(N+1)) \equiv 0, \end{aligned}$$

that, up to an irrelevant global factor $(1 - X^2)$ that never vanishes, is the Legendre equation in operatorial form.

As before, the algebraic approach starts from the commutator and the anticommutator of the operators J_{\pm}

$$[J_+, J_-] = -2(N + 1/2), \quad \{J_+, J_-\} = 2(N + 1/2)^2 + 1/2.$$

Defining the operator $J_3 := N + 1/2$, which has the same explicit form than in the Laguerre case, a $su(1, 1)$ algebra is again found:

$$[J_3, J_{\pm}] = \pm J_{\pm}, \quad [J_+, J_-] = -2J_3.$$

Moreover, the Legendre functions support the same UIR of Casimir $\mathcal{C} = -1/4$ of the Laguerre functions and the equation

$$(\mathcal{C} + 1/4) P_n(x) = (J_3^2 - \{J_+, J_-\}/2 + 1/4) P_n(x) = 0$$

together with eqs.(21) gives the Legendre equation.

A coherent state generalization of $\{P_n(x)\}$ can be also defined in this case similarly to (18)

$$P_{\alpha}(x) := e^{\hat{\xi} J_+ - \hat{\xi}^* J_-} P_0(x) = (1 - |\alpha|^2)^{1/2} \sum_{n=0}^{\infty} \alpha^n P_n(x), \quad (23)$$

where the complex numbers α and $\hat{\xi}$ are the same of the Laguerre case (18).

5 Conclusions

This paper is the first of a program attempting to give a global description of orthogonal polynomials where the different aspects –differential equations theory, Lie algebras,

Hilbert spaces and generalized Fourier series– are considered together. We discuss here orthogonal polynomials defined on the line –where technical aspects are less relevant– and we give an algebraic description of them in a global way. At the level of Hilbert spaces the Hermite polynomials allow to build a discrete basis from the square integrable functions on the line while the Laguerre and Legendre polynomials are defined on the half-line and the interval $(-1, +1)$, respectively. From the point of view of Lie algebras, the Hermite polynomials support the UIR of the Weyl-Heisenberg algebra associated to the Casimir $\mathcal{C} = 0$ and the Laguerre and Legendre polynomials are related to the fundamental UIR ($\mathcal{C} = -1/4$) of the discrete series of $su(1, 1)$ that are all explicitly constructed.

Summarizing, in all the three cases we introduce in the vector space the countable basis $\{|n\rangle\}_{n \in \mathbb{N}}$, related to the uncountable coordinate basis $\{|x\rangle\}_{x \in E \subset \mathbb{R}}$ by the appropriate orthogonal functions, on which a UIR of a non-compact Lie algebra (the Weyl-Heisenberg algebra $h(1)$ for Hermite and the $su(1, 1)$ algebra for Laguerre and Legendre) is defined. As the basis $\{|n\rangle\}$ and the associated orthogonal functions are equivalent the algebraic structure can be transferred from the $\mathcal{L}^2(E)$ space on the vector space straight to the vector space E . Moving from the Lie algebra to the UEA, all the operators defined on the space on $\mathcal{L}^2(E)$ can be generated. In view of applications it is particularly relevant the action on \mathcal{L}^2 of the groups $H(1)$ or $SU(1, 1)$. A first result related to the introduction of the algebraic structure is that coherent states defined on the the line vector space $\{|n\rangle\}$ by the Weyl-Heisenberg algebra can be transferred to the wave functions, i.e. the Hermite polynomials and, analogously, coherent Laguerre or Legendre polynomials can be defined à la Perelomov [7] starting from the properties of the appropriate vector space.

In conclusion, we believe that –combining properties of different origin like differential equations, factorization method, theory of Lie representations, Hilbert spaces, integrable functions– a better description of the orthogonal polynomials has been obtained.

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